Math 126 Midterm 3

UCB Summer 2026

August 13th, 2026

Remember that the exam is closed book. You will have 50 minutes to complete the Exam. You may use the front and back of each page. There are 3 total problems, so ensure that you have attempted all 3. Please write your full name and SID below. Good luck!

Name:

SID:

Honor Pledge:

Formulas: Characteristic ODEs:

$$\begin{aligned} x'(s) &= D_p F \\ z'(s) &= D_p F \cdot p \\ p'(s) &= -D_x F - D_z(F)p \end{aligned}$$

Divergence:

$$\int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} F \cdot \eta dS$$

D'Alembert:

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)d\tau$$

$$\mathcal{D}_{t,x} = \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} \mid x - c(t-s) \leq y \leq x + c(t-s)\}$$

Inhomogeneous Wave:

$$u(t,x) = \frac{1}{2c}\int_{\mathcal{D}_{t,x}} f(s,y)dsdy$$

Poisson:

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi}\int_D \frac{g(t-xy)}{\sqrt{1-|y|^2}}dy\right) + \frac{t}{2\pi}\int_D \frac{h(t-xy)}{\sqrt{1-|y|^2}}dy$$

Kirchoff:

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{\partial B(x,t)} g(y)dS(y)\right) + \frac{1}{4\pi t}\int_{\partial B(x,t)} h(y)dS(y)$$

Helmholtz 1D:

$$\phi_k(x) = \sin(\frac{k\pi}{l}x), \qquad k \in \mathbb{Z}$$

solve

$$-\phi(x) = \lambda\phi \qquad \phi(0) = \phi(l) = 0$$

for $\lambda = (\frac{k\pi}{l})^2$.

Helmholtz Solutions on the Torus:

$$\phi_k(x) = e^{ikx} \qquad k \in \mathbb{Z}$$

Heat Separation of Variables on the Torus:
$u(t,x) = e^{-k^2t}e^{ikx}$ for $k \in \mathbb{Z}$.
Heat Kernel:

$$H_t(x) = (4\pi t)^{-n/2}e^{-\frac{|x|^2}{4t}}$$

Heat Solution on $\mathbb{R}^n$ with initial condition $f \in C^0(\mathbb{R}^n)$:

$$u(t,x) = H_t(x) * f(x)$$

Inhomogeneous (Forcing Term $f(t,x)$) Heat solution with initial condition g:

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} H_{t-s}(x-y)f(s,y)dyds + \int_{\mathbb{R}^n} H_t(x-y)g(y)dy$$

Fourier Coefficient:

$$c_k[f] = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$$

Heat Solution on the Torus:

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k[g]e^{-k^2t}e^{ikx}$$

Poisson Kernel:

$$P_r(\theta) = \sum_{\mathbb{Z}} r^{|k|}e^{ik\theta}$$

Laplace Solution with boundary condition g:

$$u(r,\theta) = P_r(\theta) * g(\theta)$$

Mean Value Formula:

$$u(x_0) = \frac{1}{|\partial B(x_0,R)|}\int_{\partial B(x_0,R)} u(x)dS + \int_{B(x_0,R)} G_R(x-x_0)\Delta u(x)dx$$

for

$$G_R(x) = \begin{cases} \frac{1}{2\pi}\ln(\frac{r}{R}) & n=2 \\ \frac{1}{(n-2)|\partial B(0,1)|}\left[\frac{1}{R^{n-2}} - \frac{1}{r^{n-2}}\right] & n \geq 3 \end{cases}$$

Laplace Fundamental Solution:

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln(r) & n=2 \\ \frac{1}{(n-2)A_n r^{n-2}} & n \geq 3 \end{cases}$$

Representation Formulas:

$$u(y) = -\int_\Omega \Phi_y \Delta u dx + \int_{\partial\Omega}\left[\Phi_y\frac{\partial u}{\partial\eta} - u\frac{\partial\Phi_y}{\partial\eta}\right]dS$$

- For the Poisson problem with forcing term f and boundary term g:

$$u(y) = -\int_\Omega fG_y dx - \int_{\partial\Omega} g\frac{\partial G_y}{\partial\eta}dS$$

Problem 1:

This section is meant to prepare you for true-or-false questions by leaving some open-ended conceptual questions.

A.) What types of PDEs do we know that have maximum principles? What assumptions do we have on the domain? On the initial or boundary conditions?

B.) Why do we take weak solutions to PDEs? Can we have distributional solutions to PDEs? What relates weak solutions back to classical solutions in the case of the Poisson equation (can you prove this relationship)?

C.) What conditions do we need to understand the weak solution to a conservation equation with piecewise continuous initial condition?

D.) Let $f : \mathbb{T}^n \to \mathbb{R}$ have

$$\sum_{k \in \mathbb{Z}^n} |k|^8 |c_k[f]|^2 < \infty$$

How many derivatives of f can I take? How many weak derivatives? State some of the relevant results.

This page is for work

Problem 2:

Part A.) Consider the partial differential equation

$$\begin{aligned} x^2[\partial_t^2 u - \partial_{x^1}^2 u - \partial_{x^2}^2 u] - 2\partial_{x^2} u &= f \qquad\qquad (0,\infty)\times\Omega \\ u|_{t=0} &= g \\ \partial_t u|_{t=0} &= h \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

Find a weak formulation for this equation so that weak solutions would have one weak derivative (don't forget to state what space f is in).

Problem 3: (Challenge)

Consider $\mathbb{R}$. For what α is there a distribution $\phi \in \mathcal{D}'(\mathbb{R})$ such that ϕ agrees with x^α for $x \neq 0$? *Hint: Split into cases $\alpha > -2$ and $\alpha \leq -2$. In the second case, notice that if ψ is a test function with support in $(1,2)$, then $\sum_{k=1}^N 2^{-k}\psi(2^k x)$ converges in $C_c^\infty(\mathbb{R})$ as $N \to \infty$, and each partial sum has support in $(0,\infty)$. Does this cause a problem for continuity?*

Problem 4:

What is the Green's function on $B(0,1) \subset \mathbb{R}^n$ (review the method of charges)? Use this function to write the integral solution formula for Harmonic functions (it should look like a generalization of the mean value formula).

Problem 5: Can you find a pair of domains $\Omega \subset \tilde{\Omega}$ and a $u \in H^1(\Omega)$ whose extension by 0 to $\tilde{\Omega}$ is not in H^1?